Covariant Analysis of the Experimental Constraints on the Brane-world

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Some observational constraints on the brane-world based on predictions from specific models in five dimensions, have been recently reported, both on local and cosmological scales. In order to identify the origins of these constraints, the equations of motion of the brane-world are translated to the most general, model-independent (or "covariant"), formulation of the theory, based only on the Einstein-Hilbert action for the bulk geometry, the confinement of the standard gauge interactions and the exclusive probing of the extra dimensions by the gravitational field. In the case of the binary pulsar PSR1913+16, it is found that gravi-vectors and gravi-scalars do not appear in the covariant equations, but they are replaced by vector and scalar fields related to the extrinsic curvature of the brane-world. Only the latter one impose a condition on the binary pulsar orbits. A general solution for this problem is proposed, based on results from differential geometry, suggesting a stable bulk geometry, whose existence requires higher dimensions. On the cosmological scale, it is shown that the high energy inflation constraint originating from the square of the energy density term in the modified Friedman's equation is mainly due to the assumption of the reflection symmetry across the brane-world. It is shown that this symmetry is not consistent with the regularity of the brane-world. These results suggest that the two constraints can be lifted by increasing the number of extra dimensions.

I. THE CONSTRAINTS

When a theory is faced with observational constraints, it becomes necessary to verify the extent in which its predictions depend on the properties of used models. Only when those predictions are covariant in the sense of being model independent we may convince ourselves that the constraints pose a real problem to the theory in itself. This appears to be the current situation in brane-world theory in five-dimensions, where several models have been recently checked against precision astrophysical and cosmological observational constraints.

In particular, the interference of linear gravitational waves generated by the bulk geometry over the quadrupole formula for the binary pulsar PSR1913+16, predicts an error of about 20%, against an observed error of just 0.5% [1]. In the cosmological scale, different models lead to a modified Friedman's equation depending on the square of the energy density, whose effect is to produce a slowdown of the high energy inflation, in disagreement with the recent data from the WMAP/SDSS/2dF surveys [2, 3].

The purpose of this paper is find why these constraints occur, by rewriting them in the most general model-independent formulation (sometimes referred to as "covariant" formulation) and, whenever possible propose solutions. The model-independent formulation of the brane-world is characterized only by the Einstein-Hilbert

action for the bulk geometry, the confinement of the standard gauge interactions and the exclusive probing of the extra dimensions by the gravitational field at the TeV scale of energy [4]. The result is a set of bare equations of motion, to which we may add specific model properties afterwards (These are in fact well known equations which have been extensively applied to the construction of diverse models in five dimensions [5, 6, 7, 8, 9, 10, 11]).

We find that gravi-vectors and gravi-scalars do not appear in the covariant equations, but they are replaced by the components of the extrinsic curvature and its contraction, satisfying equivalent vectors and scalar equations. Therefore, using the same conditions as in [1], we obtain essentially the same graviton equations and a scalar condition which interferes with the binary pulsar orbits.

A general solution for such type of constraint is proposed, by increasing the number of extra dimensions in the covariant formulation, until the bulk becomes stable in the sense that its metric is not perturbable, regardless of what is happening with the brane-world embedded on it.

On the other hand, we have found that the primary source of the high energy inflation constraint is in most cases the Z_2 symmetry across the brane-world. We show that although this result can be implemented in the covariant formulation on a specific brane-world, in general the Z_2 symmetry is not consistent with the regularity theorems required by its perturbations.

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II. THE COVARIANT EQUATIONS OF MOTION IN ARBITRARY DIMENSIONS

The four-dimensional brane-world can be seen as the result of the motion of a 3-brane embedded in a D-dimensional bulk, $D = 4 + N$, whose geometry is defined by the Einstein-Hilbert action integral[1]

$$\int \mathcal{R} \sqrt{-\mathcal{G}} d^D v = \alpha_* \int \mathcal{L}^* \sqrt{-\mathcal{G}} d^D v \qquad (1)$$

where α_* is the bulk energy scale and $\mathcal{L}^*$ is a source Lagrangian usually describing gauge fields and ordinary matter confined to the brane-world. Taking the variation of this action with respect to the bulk metric $\mathcal{G}_{AB}$ we obtain the bulk Einstein's equations

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\mathcal{G}_{AB} = \alpha_* T^*_{AB} \qquad (2)$$

The confinement hypothesis states that the standard gauge fields and ordinary matter remain trapped in a 3-brane, or better, in the 4-dimensional manifold spanned by its motion in the bulk. On the other hand, the exclusive probing of the extra dimensions by TeV gravitons say that the geometry of that manifold present oscillation modes at that energy scale. Thus, these two postulates require a submanifold structure, the brane-world, which remains always embedded in the bulk.

The embedding of a manifold into another can be realized in many different ways and the choice of one or another depends on what it is supposed to do. The action principle (1) suggests that the four-dimensional gravitational field is induced by that of the bulk, and the simplest realization of such induction is through a local and isometric embedding.

A very common simplification consists in assuming that the embedding functions are analytic in the sense that they are representable by convergent positive power series. This type of embedding is is useful to prove theorems in mathematics, but it bypasses some of the differentiable properties required by the dynamics of the brane-world. Thus, except in some particular instances, the analytic embedding is not suitable for the brane-world whose geometry represents a high energy field (at the least at the TeV scale). The more general differentiable embedding, obtained via differentiable perturbations of a given background, is briefly reviewed below, mostly extracted from the classic literature on this subject [12, 13, 14].

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[1] Notation: Curly curvature components $\mathcal{R}_{...}$ refer to the bulk, while straight curvature components $R_{...}$ refer to the brane-world. Capital Latin indices run from 1 to D. Small case Latin indices refer to the extra dimensions only, running from 5 to D. All Greek indices refer to the brane-world, counting from 1 to 4. An overbar denotes an object of a fixed background brane-world geometry. The semicolon denotes the covariant derivative with respect to the brane-world metric $g_{\mu\nu}$.

The differentiable embedding of a given manifold $\bar{V}_4$ with metric $\bar{g}_{\mu\nu}$ in an arbitrary bulk V_D, $D = 4 + N$, with metric $\mathcal{G}_{AB}$, is given by D differentiable maps $\bar{\mathcal{X}}^A : \bar{V}_4 \to V_D$, satisfying the (isometric) embedding equations

$$\bar{\mathcal{X}}^A_{,\alpha}\bar{\mathcal{X}}^B_{,\beta}\mathcal{G}_{AB} = \bar{g}_{\alpha\beta}, \ \bar{\mathcal{X}}^A_{,\alpha}\bar{\eta}^B_a\mathcal{G}_{AB} = 0, \ \bar{\eta}^A_a\bar{\eta}^B_b\mathcal{G}_{AB} = \epsilon_a\delta_{ab} \ (3)$$

where $\bar{\eta}_a$ denotes the components of the $N = D - 4$ orthogonal vectors normal to $\bar{V}_4$, and $\epsilon_a = \pm 1$ correspond to the two possible signatures of each extra dimension. Once we have the embedding of that particular $\bar{V}_4$, we may deform (or perturb) it along an arbitrary direction ζ in the bulk, given by the Lie derivative of the embedding coordinates

$$\mathcal{Z}^A = \bar{X}^A + (\pounds_\zeta \bar{\mathcal{X}})^A \qquad (4)$$

To avoid possible coordinate gauges which could trigger false perturbations, as in [13] we consider the deformations along the unit normals η_a, parameterized by the extra coordinates y^a. In this case, the components of the deformed embedding functions $\mathcal{Z}^A = \bar{X}^A + y^a\eta^A_a$ must satisfy embedding equations similar to (3), with the difference that now they depend on y^a. Using (3) and (4), we obtain the geometry of the perturbed manifold

$$g_{\mu\nu}(x,y) = \mathcal{Z}^A_{,\mu}\mathcal{Z}^B_{,\nu}\mathcal{G}_{AB} = \bar{g}_{\mu\nu} - 2y^a\bar{k}_{\mu\nu a} +$$
$$y^a y^b[\bar{g}^{\alpha\beta}\bar{k}_{\mu\alpha a}\bar{k}_{\nu\beta b} + g^{cd}\bar{A}_{\mu ca}\bar{A}_{\nu db}], \quad (5)$$

$$g_{\mu a}(x,y) = \mathcal{Z}^A_{,\mu}\eta^B_a\mathcal{G}_{AB} = y^a A_{\mu ab}, \qquad (6)$$

$$g_{ab}(x,y) = \eta^A_a\eta^B_b\mathcal{G}_{AB} = \epsilon_a\delta_{\mu a}, \epsilon_a = \pm 1, g^{ab}g_{bc} = \delta^a_c \ (7)$$

where the first equals signs show that these are the components of the bulk metric $\mathcal{G}_{AB}$ evaluated in the embedding vielbein $\{\mathcal{Z}^A_{,\mu}, \eta^A_a\}$

In addition to the metric components we have also the extrinsic curvature and the "torsion" vector, respectively given by

$$k_{\mu\nu a}(x,y) = -\eta^A_{a,\mu}\mathcal{Z}^B_{,\nu}\mathcal{G}_{AB} = \qquad (8)$$
$$\bar{k}_{\mu\nu a} - y^b\bar{g}^{\alpha\beta}\bar{k}_{\mu\alpha a}\bar{k}_{\nu\beta b} - g^{cd}y^b\bar{A}_{\mu ca}\bar{A}_{\nu db},$$

$$A_{\mu ab}(x,y) = \eta^A_{a,\mu}\eta^B_b\mathcal{G}_{AB} = \bar{A}_{\mu ab}(x) \qquad (9)$$

Notice that (5) and (8) imply that the extrinsic curvature also propagates in the bulk, according to York's relation (extended to the extra variables y^a):

$$k_{\mu\nu a} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y^a} \qquad (10)$$

However, from (9) it follows that $A_{\mu ab}$ does not propagate at all in the bulk. Finally, defining $h^{AB} = g^{\mu\nu}Z^A_{,\mu}Z^B_{,\nu}$, with inverse $h_{AB} = \mathcal{G}_{AM}\mathcal{G}_{BN}h^{MN}$, we obtain from (5)-(7)

$$h^{AB} = \mathcal{G}^{AB} - g^{ab}\eta^A_a\eta^B_b \qquad (11)$$

In order to guarantee that the perturbed manifold remains embedded in the same bulk, the Riemann tensor $\mathcal{R}_{ABCD}$ must be independent of the hypersurface on

which its components are expressed. Therefore, using the vielbein $\{\mathcal{Z}^A_{,\mu}, \eta^A_a\}$ defined by the perturbed hypersurface, the components of that tensor give the required condition. After eliminating the redundant expressions, the remaining equations are the well known Gauss, Codazzi and Ricci equations:

$$\mathcal{R}_{ABCD}\mathcal{Z}^A_{,\mu}\mathcal{Z}^B_{,\nu}\mathcal{Z}^C_{,\rho}\mathcal{Z}^D_{,\sigma} = R_{\mu\nu\rho\sigma} - 2g^{cd}k_{\mu[\rho c}k_{\sigma]\nu d} \quad (12)$$

$$\mathcal{R}_{ABCD}\mathcal{Z}^A_{,\mu}\eta^B_a\mathcal{Z}^C_{,\nu}\mathcal{Z}^D_{,\rho} = 2k_{\mu[\nu a;\rho]} - 2g^{cd}A_{[\rho ca}k_{\mu\nu]d} \quad (13)$$

$$\mathcal{R}_{ABCD}\eta^A_a\eta^B_b\mathcal{Z}^C_{,\mu}\mathcal{Z}^D_{,\nu} = -2A_{[\mu ab;\nu]} - 2g^{cd}A_{[\mu ca}A_{\nu]db}$$
$$- 2g^{\alpha\beta}k_{[\mu\alpha a}k_{\nu]\beta b} \quad (14)$$

where brackets apply to the adjoining indices only.

It is clear that the equations of motion of the brane-world must also be compatible with those equations. In fact they can be derived from the contraction of (12) with $g^{\mu\nu}$. After using(11) we obtain

$$\mathcal{R}_{AB}Z^A_{,\mu}Z^B_{,\nu} = R_{\mu\nu} - g^{cd}(g^{\alpha\beta}k_{\mu\alpha c}k_{\nu\beta d} - H_c k_{\mu\nu d}) -$$
$$g^{ab}\mathcal{R}_{ABCD}\eta^A_a Z^B_{,\mu}Z^C_{,\nu}\eta^D_b \quad (15)$$

where we have denoted $H_a = g^{\mu\nu}k_{\mu\nu a}$. A further contraction with $g^{\mu\nu}$ gives the Ricci scalar

$$\mathcal{R} = R - (K^2 - H^2) + 2g^{ab}\mathcal{R}_{AB}\eta^A_a\eta^B_b -$$
$$g^{ad}g^{bc}\mathcal{R}_{ABCD}\eta^A_a\eta^B_b\eta^C_c\eta^D_d \quad (16)$$

where $K^2 = g^{ab}k^{\mu\nu}_{a}k_{\mu\nu b}$ and $H^2 = g^{ab}H_a H_b$. From (16) we may write the same Einstein-Hilbert action (1), but now expressed in terms of the intrinsic and extrinsic brane-world geometry

$$\int \mathcal{R}\sqrt{-\mathcal{G}}d^D v = \int [R - K^2 + H^2]\sqrt{-\mathcal{G}}d^D v +$$
$$\int [2g^{ab}\mathcal{R}_{AB}\eta^A_a\eta^B_b - g^{ad}g^{bc}\mathcal{R}_{ABCD}\eta^A_a\eta^B_b\eta^C_c\eta^D_d]\sqrt{-\mathcal{G}}d^D v$$
$$= \alpha_* \int \mathcal{L}^* d^D v \quad (17)$$

If wished, at the level of the variational principle, we may add terms such as boundaries and cosmological constant.

Taking the variation of (17) with respect to the separate metric components $g_{\mu\nu}$, $g_{\mu a}$ and to g_{ab}, and denoting by $T^*_{\mu\nu} = T^*_{AB}\mathcal{Z}^A_{,\mu}\mathcal{Z}^B_{,\nu}$, $T^*_{\mu a} = T^*_{AB}\mathcal{Z}^A_{,\mu}\eta^B_a$ and $T^*_{ab} = T^*_{AB}\eta^A_a\eta^B_b$ the corresponding energy-momentum tensor components derived from $\mathcal{L}^*$, we obtain the covariant equations of motion consistent with (15) and (16)

$$(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - Q_{\mu\nu} - g^{ab}\mathcal{R}_{AB}\eta^A_a\eta^B_b g_{\mu\nu} +$$
$$(W_{\mu\nu} - \frac{1}{2}Wg_{\mu\nu}) = \alpha_* T^*_{\mu\nu}, \quad (18)$$

$$k^\rho_{\mu a;\rho} - H_{a,\mu} - \frac{1}{2}(R - K^2 + H^2)g_{a\mu} + A_{\rho ca}k^{\rho c}_\mu - A_{\mu ca}H^c -$$
$$W_{\mu a} + g^{mn}\mathcal{R}_{AB}\eta^A_m\eta^B_n g_{\mu a} = \alpha_* T^*_{\mu a} \quad (19)$$

$$R - K^2 + H^2 + \frac{N-1}{2}\mathcal{R} + \frac{1}{2}W = \alpha_* T^* \quad (20)$$

where we have denoted

$$Q_{\mu\nu} = g^{ab}k^\rho_{\mu a}k_{\rho\nu b} - H^a k_{\mu\nu a} - \frac{1}{2}(K^2 - H^2)g_{\mu\nu} \quad (21)$$

$$W_{\mu\nu} = g^{ad}\mathcal{R}_{ABCD}\eta^A_a\mathcal{Z}^B_{,\mu}\mathcal{Z}^C_{,\nu}\eta^D_d \quad (22)$$

$$W_{\mu a} = g^{mn}\mathcal{R}_{ABCD}\eta^A_a\eta^B_m\mathcal{Z}^A_{,\mu}\eta^D_n \quad (23)$$

$$W = g^{ad}g^{bc}\mathcal{R}_{ABCD}\eta^A_a\eta^B_b\eta^C_c\eta^D_d \quad (24)$$

Notice that when all extrinsic properties are removed in (18) we recover the usual Einstein's equations in four dimensions with the appropriate value of α_*. However, with the embedding the presence of the extrinsic curvature components is unavoidable. In particular, the tensor $Q_{\mu\nu}$ is quadratic in the extrinsic curvature and it is conserved in the sense that $Q^{\mu\nu}_{;\nu} = 0$, as it can be directly verified [15].

Notice also that the above equations are equivalent to (2). The difference is that the solutions of (18)-(20) represent a perturbation generated family of brane-worlds, whose embedding in the bulk is already built in these equations, while in the case of (2), the embeddings of the perturbations need to be checked afterwards. This difference is due to the distinct choice of dynamical variables for the same action. In (2) the dynamical variables are the components of the bulk metric $\mathcal{G}_{AB}$ chosen after a metric ansatz, while in (18)-(20) the components of the same metric (5)-(7), are obtained in the embedding frame. This may indicate that the gravi-scalar and gravi-vectors depend on the basic brane-world in which they are written. Under a perturbation of that brane-world these quantities may change. In this respect it is enlightening to compare with another choice of dynamical variables $(\mathcal{Z}^A)$ for the same action, made in the past, producing a weaker set of equations as compared with (2) [17].

III. CONSTRAINING THE FIVE-DIMENSIONAL BULK

Since all reported constraints refer to five dimensional models, in this section we restrict the previous analysis to $D = 5$, noting that in this case $A_{\mu\nu a} = 0$ and $W = 0$. For notational simplicity denote $k_{\mu\nu 5} = k_{\mu\nu}$. Also, to simplify our arguments, we fix the bulk signature to be $(4,1)$ (That is $g^{55} = 1$). Then, the covariant equations (18)-(20) simplify to

$$(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - Q_{\mu\nu} - R_{AB}\eta^A\eta^B g_{\mu\nu} + W_{\mu\nu} = \alpha_* T^*_{\mu\nu}, \quad (25)$$

$$k^\rho_{\mu;\rho} - H_{,\mu} = \alpha_* T^*_{\mu 5}, \quad (26)$$

$$R - K^2 + H^2 = -2\alpha_* T^*_{55} \quad (27)$$

In the following we apply these equations to the analysis of the binary pulsar and the high energy inflation constraints.

1- Constraints generated by bulk gravitational waves

The model used in [1] is based on a fairly general metric ansatz defined in a cylindrical bulk with an oscillating radius, but without implementing the Z_2 symmetry. Considering the linear expansion of the bulk metric

$$\mathcal{G}_{AB} = \eta_{AB} + \gamma_{AB} \tag{28}$$

and applying the corresponding de Donder Gauge, we obtain the wave equations $\Box^2\Psi_{AB} = \alpha_* T^*_{AB}$. Using a coordinate system in which these equations split as

$$\Box^2\Psi_{\mu\nu} = \alpha_* T^*_{\mu\nu} \quad \text{gravitons} \tag{29}$$

$$\Box^2\Psi_{\mu 5} = \alpha_* T^*_{\mu 5} \quad \text{gravi-vector} \tag{30}$$

$$\Box^2\Psi_{55} = \alpha_* T^*_{55} \quad \text{gravi-scalar} \tag{31}$$

By following a perturbative approach, it was found that the gravi-vector does not produce any appreciable consequence on the orbits of the binary pulsar PSR1913+16, but the gravi-scalar induce a slowdown of its period by -2.87×10^{-12}, instead of the known experimental value $-(2.408 \pm 0.001) \times 10^{-12}$ [1]. Although these results are claimed to be coordinate gauge independent, it is not clear the extent in which they depend on the chosen bulk.

To see how the same problem translates to the covariant formulation, consider the same linear perturbation of the bulk geometry. It is a simple matter to see from (5) that this perturbation is transferred to the brane-world metric as

$$g_{\mu\nu}(x,y) = \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} = \mathcal{Z}^A_{,\mu}\mathcal{Z}^B_{,\nu}\gamma_{AB}$$

and from (10) we obtain $k_{\mu\nu} = -\partial\gamma_{\mu\nu}/2\partial y$. Since $Q_{\mu\nu}$ is quadratic in $k_{\mu\nu}$, it follows that such term is negligible in the presence of linear terms on $\gamma_{\mu\nu}$. Consequently, in the same cylinder bulk and using $T_{\mu\nu} = \bar{T}_{\mu\nu}\delta(y)$, $T_{\mu 5} = T_{55} = 0$, equation (25) becomes $\Box^2\Psi_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu}\delta(y)$, which is essentially the same graviton equation (29).

On the other hand, instead of the equations (29), (30), we have the equations (26) which corresponds to the trace of Codazzi's equations and this will be an identity in a bulk with cylindrical topology. Finally, in the same approximation we obtain that $H^2 = 0$ and $K^2 = 0$, so that equation (27) in the same de Donder gauge is equivalent to $g^{\mu\nu}\Box^2\Psi_{\mu\nu} = 0$, implying that $T_{\mu\nu}$ must be trace-free.

Therefore, we obtain from the covariant equations of motion a set of equations which is equivalent to the equations in [1], suggesting that the bulk generated gravitational waves in general imply in a constraint to the binary pulsar, but its effectiveness must be revalued.

It is interesting to note that the massive modes were not taken into consideration in [1]. However, when the extra dimension is compact, the tangent components of the wave functions can be always harmonically expanded as

$$\Psi_{\mu\nu} = \sum_n \beta^{(n)}_{\mu\nu} \, e^{\frac{in\pi y}{\ell}}$$

Thus, denoting $M_n = n^2\pi^2/\ell^2$ and the n-mode by $\Psi^{(n)}_{\mu\nu}$, the wave equation can be written as

$$\Box^2\Psi_{\mu\nu} = \sum_n (\eta^{\alpha\beta}\partial_\alpha\partial_\beta - M_n^2)\Psi^{(n)}_{\mu\nu} = 8\pi GT_{\mu\nu}$$

Unless we restrict ourselves to the zero-mode, a four-dimensional observer naturally interprets M_n as a mass attached to the n-mode $\Psi^{(n)}_{\mu\nu}$. As it has been suggested in another contexts (as for example affecting the bending of light rays from distant sources by the Sun [16]), depending on the size of ℓ, such masses may contribute to local gravity effects on the brane-world, including the binary pulsar.

The existence of the binary pulsar constraint suggests that the bulk geometry should be algebraically special, of a non-radiative type. However, it is hard to explain such hypothesis on theoretical grounds. More realistically, the binary pulsar constraint can be seen as an indication that the dimension of the bulk cannot be fixed to five. Indeed, as a manifold the bulk can also be embedded in a larger bulk. Thus, supposing that we have initially a five-dimensional flat bulk whose geometry is perturbed, producing a new non-flat five-dimensional bulk, also containing the original brane-world. The perturbed non-flat bulk can be embedded in, say, a six-dimensional flat bulk. If for some physical action the geometry of this six-dimensional bulk is again perturbed, then it can be re-embedded in an even higher-dimensional flat bulk. Such re-embedding process can be repeated over and over, until reaching a sufficiently higher-dimensional flat bulk whose geometry is flat and stable in the sense that it does not change regardless of the dynamical state of the embedded brane-world. According with such view, the binary pulsar is suggesting that $D = 5$ is not sufficient.

This flat-stable bulk can be considered as the analogous to the ground state of Kaluza-Klein theory, represented by the D-dimensional Minkowski space solution of Einstein's equations. However, this analogy is only partial because Kaluza-Klein theory is based on the product topology $V_4 \times B_N$ and not on a dynamical submanifold structure like in the brane-world case. Nonetheless, V_4 can be seen as locally embedded in the product space obeying the same Einstein-Hilbert dynamics, leading to some variations of the Kaluza-Klein theory which closely resembles the brane-world theory [18, 19, 20, 21].

In the general case, the existence of such flat-stable bulk was established originally by Nash and later on generalized to non positive metrics by Greene [13, 14], producing the general expression for the bulk dimension: $D = n(n + 3)/2$. For a 3-brane this gives $D = 9$ and for the brane-world we obtain $D = 14$. It is important to realize that this has nothing to do with supergravity which would impose a $D = 11$ limit. Yet, it has been noted that a 14-dimensional bulk with signature $(13, 1)$ can also embed some interesting super-algebras [22]. Also it should be remenbered that the use of the extra embedding dimensions as a possible generator of gauge symmetries has

been proposed long ago [23, 24], suggesting an $SO(10)$ based GUT model on the the brane-world.

2 - <u>High Energy Inflation Constraint</u>

Here we have a different problem resulting from the presence of the ρ^2 term in the brane-world modified Friedman's equation. Accordingly, the presence of such term implies in a slowing down of the high energy inflation, inconsistently with the anisotropy data from the WMAP/SDSS/2dF experiments [2, 3].

In order to understand how the ρ^2 term appears in Friedman's equation consider again Einstein's equations for the bulk, now written as

$$\mathcal{R}_{AB} = \alpha_*(T^*_{AB} - \frac{1}{3}T^*g_{AB}) \qquad (32)$$

Using a Gaussian normal coordinates on the brane-world, in which the metric components are $g_{\mu\nu}$, $g_{\mu 5} = 0$ and $g_{55} = 1$, the tangent components $\mathcal{R}_{\mu\nu}$ of the above equations are (after using (10))

$$R_{\mu\nu} - \frac{\partial k_{\mu\nu}}{\partial y} - 2k^\rho_\mu k_{\rho\nu} + hk_{\mu\nu} = \alpha_*(T^*_{\mu\nu} - \frac{1}{3}T^*g_{\mu\nu})\,(33)$$

Taking the brane-world as a boundary, separating two regions of the bulk, labeled by $+$ and $-$ respectively, the difference between the above components, calculated on each side of the brane-world for $y \to 0$ is zero because there is no real distinction of the Riemann geometry of the bulk as seen from each side. This situation changes when the Z_2 symmetry is assumed across the brane-world $\bar{V}_4$, so that the brane acts as a mirror. In this case, an object that senses the extra dimension in one side is mirrored by $\bar{V}_4$ to its image. This is the case of the extrinsic curvature which measure the tangent component of the variation of the normal vector η, when its foot is displaced on the brane-world. From the mirror image of this variation we obtain $k^+_{\mu\nu} = -k^-_{\mu\nu}$. Using the mean value theorem for the derivative of $k_{\mu\nu}$ with respect to y, we also find that (with $k^+_{\mu\nu} = k_{\mu\nu}$)

$$-(\frac{\partial k_{\mu\nu}}{\partial y})^+ + (\frac{\partial k_{\mu\nu}}{\partial y})^- = -2\frac{k_{\mu\nu}}{y}$$

Denoting $[X] = X^+ - X^-$, and $X = \bar{X}(x)\delta(y)$, under the Z_2 symmetry it follows that in the limit $y \to 0$

$$|y|[X] = \int_{-y}^{y} |y|\bar{X}\delta'(y)dy + \int_{-y}^{y} \frac{y}{|y|}\bar{X}\delta(y)dy = 2\bar{X}$$

Replacing these expressions in the difference of the expression (33) calculated in both sides, for $X = \bar{T}_{\mu\nu}$, we obtain at $y = 0$ the Israel-Lanczos condition

$$\bar{k}_{\mu\nu} = -\alpha_*(\bar{T}_{\mu\nu} - \frac{1}{3}\bar{T}\bar{g}_{\mu\nu}) \qquad (34)$$

Therefore, this condition follows from Einstein's bulk equations (2), plus the Z_2 symmetry, plus the delta function confinement of $T^*_{\mu\nu}$.

The Z_2 symmetry across the brane-world was originally motivated by the Horava-Witten theory to compactify the 11-dimensional M-theory to the product topology $V_{10} \times S^1/Z_2$. However, when the same principle is transposed to the brane-world theory based on the Einstein-Hilbert action in five dimensions, the bulk becomes orbifold compactified to $V_4 \times S^1/Z_2$ with the identification $-y \to y$ [25, 26, 27, 28]. However, S^1/Z_2 is not a manifold and some of the conditions required for the differentiable embedding fail to apply. Actually, under the Z_2 symmetry, all perturbations of the brane-world have a mirror perturbation on the opposite side of the background, with the derivatives of the normal having opposite signs. Consequently, the regularity of the embedding functions is not generally defined and the mentioned theorems of Nash and Greene, which depend on extended regularity, fail to apply. In other words, the implementation of the Z_2 symmetry is not completely consistent with covariant formulation of the brane-world. However, if we restrict this to the background brane-world $\bar{V}_4$, as it is the case of (34), then it does not really matter because this background was assumed to be embedded in the first place.

The implications of this symmetry, or of (34), to the high energy inflation is as follows: Taking $\bar{T}_{\mu\nu}$ as the energy-momentum tensor of the confined perfect fluid in (34) and replacing $\bar{k}_{\mu\nu}$ in the expression of $Q_{\mu\nu}$ in (25), then Friedman's equation becomes modified by the addition of a squared energy density ρ^2 term [5, 29]. Therefore, the high energy constraint remains valid in the background geometry only and it is suggesting that the Z_2 symmetry is not consistent with the perturbations of that background along the extra dimensions. It is possible to replace the Z_2 symmetry by alternative conditions applied to the geometry of the brane-world. In these cases the effects of the alternative conditions on the differentiable structure need to be examined separately [28, 30].

Concluding Remarks

The binary pulsar PSR1913+16 has proven to be a valuable tool to test alternative gravitational theories [31]. Therefore, it should be applied as a test to the brane-world proposal, as it was done in [1]. However, to be certain, the evaluation must be done in a model-independent fashion, as long as we have a consensus on what is meant by model-independent. In the present note we have derived the equations of motion for a general brane-world in an arbitrary dimensions, based only on the Einstein-Hilbert action principle for the bulk geometry, the confinement hypothesis and the exclusive probing of the extra dimensions by the four-dimensional gravitational field, leaving aside any specific model property. The probing energy can be fixed at the TeV scale, but here this was generally assigned to a constant α_*.

Taking the dynamical variables as the bulk metric components in the embedding frame, we found the equations of motion (18)-(20) which describe an embedded brane-world in the bulk, under the very reasonable conditions

that their embedding functions remain differentiable and regular. It follows from these equations that the five-dimensional bulk may not be sufficient to embed all possible configurations of a dynamically evolving brane-world. Yet, models defined in the five-dimensional bulks have become so popular in the past five years that it almost become a synonymous of the brane-world theory. The binary pulsar constraint corroborate that such limitation on the bulk dimension cannot be maintained.

This conclusion is supported by the high energy inflation constraint. Indeed, we have shown that the Z_2 symmetry adopted in the most of the five-dimensional brane-world cosmologies is the primary source of the cause of the high energy inflation constraint. As it happens, this symmetry implies that all perturbations of a brane-world has a mirror image with respect to $y = 0$ and this implies that the tangent components of the normal vector derivatives have opposite directions. When this is used as an orbifold compactification, the regularity of the embedding functions becomes undefined and consequently the differentiable embeddings are compromised. The high energy inflation constraint is telling us that the Z_2 symmetry and the orbifold compactification must not be present on the brane-world. Therefore, the two constraints suggests by different ways that the brane-world program requires more than five dimensions.

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